Structural Dynamics of Base Isolated Buildings

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This paper describes a mathematical model for the seismic elastic response of base isolated buildings as a function of the superstructure dynamic properties when this is assumed fixed at the base, and of the same superstructure when this is assumed to behave as a rigid body with a single degree of freedom when mounted on base isolators. The model includes two degrees of freedom, one each, for the superstructure and the base.

The expressions derived in this paper provide the period and damping of the isolated building, the magnitude of the seismic story forces, the base shear, and the base displacement. The expressions are simple and can be readily cast into building code format.

The derived expressions are compared to the 1994 Uniform Building Code (UBC), and a case study suggests an advantage of using the recommendations proposed in this paper versus the procedure currently recommended by the UBC.

INTRODUCTION

Research on the use of laminated steel-rubber isolators for structural seismic protection began in the early seventies at the Laboratoire de Mécanique et d'Acoustique of the Centre National de la Recherche Scientifique (CNRS) in Marseille, France. A number of structures in France and California , including nuclear facilities, have to date been provided with the isolation system developed therein. Since that period, researchers in Europe, Japan, New Zealand, and the United States have developed their own types of base isolation systems which are being implemented in a variety of projects.

However, despite the solid theoretical background of the base isolation principles, and the increasing field evidence of efficient behavior during actual earthquake occurrences, base isolation as a means to protect structures is seldom used. A partial reason for this situation was the lack of regulations. This began to be addressed with the inclusion into codes such as the Uniform Building Code [1] of provisions for guiding the use of this leading technique. Following this lead, the present paper intends to provide a number of expressions that furnish the period and damping of the isolated building, the story forces, the base shear, and the base displacement. These expressions are derived by straightforward application of structural dynamics principles. They are simple to use and can be readily formatted into future code type equations for the design of base isolated buildings.

THEORETICAL DEVELOPMENT

Early studies, [4], [7], indicated that the seismic response of base isolated structures was confined mostly to the fundamental mode. These results suggest that a two degree of freedom (D.O.F.) simplified model could be used to predict the response of such systems. To this effect let us consider figure 1.a, which shows a base isolated building assimilated to the conceptual model shown in figure 1.b, wherein the superstructure is characterized by its generalized mass M_S , its generalized damping C_S, and its generalized stiffness K_S. We assume that the superstructure would respond in its first mode when fixed at its base, and that the above values pertain to this mode. The base isolators are considered to be axially stiff, but very flexible horizontally. Therefore, the base is characterized by its mass M_b , the total damping provided by the isolation system C_b, and the total horizontal stiffness of the isolators K_b. As shown in Appendix A, the equation of motion for the model depicted in figure 1.b, when excited by a horizontal earthquake with acceleration *a(t),* is approximated by:

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) = -Mra(t) \tag{1}$$

where *r* is a vector of ones, *y(t)* is the vector of horizontal displacements relative to the ground. The base and superstructure components are y_b and y_S respectively. *M*, *C*, and *K* are the mass, damping, and lateral stiffness matrices of the system respectively, and are given by:

$$M = \begin{bmatrix} M_S & 0 \\ 0 & M_b \end{bmatrix} \tag{1.a}$$

$$K = \begin{bmatrix} K_S & -K_S \\ -K_S & K_S + K_b \end{bmatrix} \tag{1.b}$$

$$C = \begin{bmatrix} C_S & -C_S \\ -C_S & C_S + C_b \end{bmatrix} \tag{1.c}$$

If we assume that the system under consideration is classically, and lightly, damped, the frequencies and mode shapes may be obtained by solving the eigen problem:

$$\left[K - w^2 M\right] X = 0 \tag{2}$$

where w is the circular frequency of the system for the corresponding mode shape *X*. Prior to solving this equation, let us define the following variables:

$$w_I^2 = \frac{K_b}{M_b + M_S} \tag{3.a}$$

$$w_f^2 = \frac{K_S}{M_S} \tag{3.b}$$

$$\Omega = \frac{w_I}{w_f} \tag{3.c}$$

$$m = \frac{M_b}{M_S} \tag{3.d}$$

Equation (3.a) provides the circular frequency for a perfectly rigid superstructure mounted on isolators and having a single horizontal degree of freedom (depicted in figure 2.b.). Equation (3.b) provides the circular frequency for the superstructure in its fixed base state (depicted in figure 2.a.). Equation (3.c) represents a frequential ratio, usually small as well isolated buildings have high values of fixed base frequency in relation to their isolated single degree of freedom frequency. Equation (3.d) is a ratio between the base mass and the generalized mass of the superstructure. The eigenvalues from equation (2) are:

$$\begin{matrix} w_1^2 \\ w_2^2 \end{matrix} = w_f^2 \frac{(1+m)(1+\Omega^2)}{2m}\left[1 \mp \sqrt{1 - \frac{4m\Omega^2}{(1+m)(1+\Omega^2)^2}}\right]$$

The terms under the radical may be developed by binomial series, so the fundamental frequency is finally written:

$$w_1 = \frac{w_I}{\sqrt{1+\Omega^2}} \tag{4}$$

Since the frequential ratio for the structures under study is small, equation (4) indicates that the fundamental frequency of base isolated buildings is close to the frequency of single degree of freedom systems constituted by rigid superstructures mounted on horizontally flexible base isolators. For sufficiently low values of the horizontal isolation system stiffness K_b, the first mode frequency may be placed in the descending branch of typical acceleration spectra. This is the intent of base isolation; to decouple the structure from ground motions by detuning it with respect to the high frequency contents of the seismic acceleration. The fundamental mode is obtained by replacing equation (4) into equation (2), which results in:

$$X_1 = \begin{bmatrix} 1+\Omega^2 & 1 \end{bmatrix}^T \tag{5}$$

where the base modal component has been normalized to one. Equation (5), given the small frequential ratio, shows that the isolated building tends to behave globally as a single degree of freedom system undergoing a uniform translational motion with reduced relative interstory displacement. Similarly, the second mode frequency may be written as:

$$w_2 = w_f \sqrt{\frac{(1+m)(1+\Omega^2)}{m}} \tag{6}$$

Equation (6) shows that the addition of the base mass increases the fixed base frequency, in particular for small values of the mass ratio. The corresponding second mode is:

$$X_2 = \begin{bmatrix} -\dfrac{m}{1+\Omega^2} & 1 \end{bmatrix}^T \tag{7}$$

where the base component has also been normalized to one. Let us recall that the modal participation factors are written:

$$g_i = \frac{X_i^T Mr}{X_i^T MX_i} \tag{8.a}$$

where the subscripts denote the order of the mode under consideration. After replacement of the appropriate values we obtain for the first mode:

$$g_1 = \frac{m+(1+\Omega^2)}{m+(1+\Omega^2)^2} \tag{8.b}$$

Equation (8.b) shows that the participation factor for the first mode, given the small frequential ratio, approaches one, which is indicative of single degree of freedom behavior. The participation factor for the second mode is written:

$$g_2 = \frac{\Omega^2(1+\Omega^2)}{m+(1+\Omega^2)^2} \tag{8.c}$$

Equation (8.c) shows that the participation factor for the second mode is very small, since it is affected by the square of the frequential ratio. Therefore, even if the frequency given by equation (6) falls in the range of high spectral acceleration, the smallness of the participation factor ensures that the second mode is not highly excited by the ground motion. We proceed now to determine the modal damping coefficient for the fundamental mode, z_1 . We use the expression for the generalized damping:

$$X_1^T C X_1 = 2V_1 w_1 X_1^T M X_1 \tag{9}$$

After the appropriate replacements and recalling that:

$$C_S = 2V_S w_f M_S$$

$$C_b = 2V_b w_I (M_S + M_b)$$

we obtain for sufficiently small frequential ratios:

$$V_1 = V_b + V_s \frac{\Omega^3}{1+m} \tag{10}$$

where z_b and z_S are the isolation system damping coefficient and the superstructure damping coefficient, respectively. Equation (10) shows that the overall damping is provided mostly by the isolation system, as the cube of the frequential ratio is very small. Physically this corresponds to a quasi-rigid behavior of a building with reduced interstory displacements, and with the bulk of the global displacement happening across the isolation interface. The equation also shows that in order to benefit from damping, the isolation system has to furnish at least equal or better energy dissipation characteristics than those provided by the superstructure. Finally, equation (10) indicates that adding energy dissipators to the superstructure of base isolated buildings is redundant. It is interesting to note that when making m equal to zero in equation (10), we obtain the same equation proposed by ATC 3-06, [2], for systems with soil-structure interaction. Similarly for the second mode, for sufficiently small frequential ratios, the damping coefficient z_2 is given by:

$$V_2 = V_b \frac{\Omega}{\sqrt{m(1+m)}} + V_S \sqrt{\frac{1+m}{m}} \tag{11}$$

Equation (11) indicates that both the isolation system damping, and the superstructure damping are important for the second mode. In particular, for very small values of m the superstructure damping may be significantly enhanced.

DYNAMICS OF PARTIALLY RESTRAINED 2 D.O.F. STRUCTURES

It is pertinent at this point, to assume that the link between the base of the building shown in figure 1.a and the ground, is deleted. If in addition we assume that there is no friction between the base mass and the ground surface, the building is horizontally unrestrained. In view that the vertical motion is precluded by the ground, the building will present one rigid body mode of vibration, corresponding to the unrestrained global degree of freedom. Since friction does not exists, the base shear for either of these two modes has to be zero, that is:

$$V_i = -w_i^2 (M_S X_S + M_b X_b) e^{iw_i t} = 0$$

where the subscript *i* indicates the order of the mode considered. This equation is satisfied when:

$$w_{1PR} = 0 \tag{12}$$

and also when:

$$X_{SPR} = -m X_{bPR} \tag{13}$$

were the subscript *PR* signifies partially restrained behavior. We need to rewrite equation (2) as the quotient of Rayleigh:

$$w_i^2 = \frac{X_i^T K X_i}{X_i^T M X_i}$$

If we make K_b equal to zero, from the quotient of Rayleigh we find that for a null frequency the corresponding fundamental mode shape, normalized to one, is:

$$X_{1PR} = [1 \qquad 1]^T \tag{14}$$

Equation (14) corresponds to rigid mode behavior. Likewise, from equation (13) with the base component normalized to one, i.e. for a second mode shape of:

$$X_{2PR} = [-m \qquad 1]^T \tag{15}$$

we find that the corresponding frequency is:

$$w_{2PR} = w_f \sqrt{\frac{1+m}{m}} \tag{16}$$

The participation factors are:

$$g_{1PR} = 1 \tag{17.a}$$

and:

$$g_{2PR} = 0 \tag{17.b}$$

And finally, the damping coefficient for the second mode is:

$$V_2 = V_S \sqrt{\frac{1+m}{m}} \tag{18}$$

We see that if in the expressions previously derived for base isolated structures, we let the frequential ratio approach zero, we converge to the equations derived for the partially restrained behavior. The fundamental frequency and corresponding damping, equations (4) and (10) can not be retrieved, since the partially restrained first frequency is zero, whereas the derivation that led to the last two parameters presumed a vibrational state.

The equations derived for two degree of freedom partially restrained structures indicate that during seismic events such structures would remain in place, with zero distortions, while the ground moves horizontally underneath them. The participation of the second mode is nil. These observations can be extended to multistory partially restrained buildings. Due to the frictionless support none of the modes will induce shear at the base. In addition, as we will show below, given that the rigid mode shape consists of a uniform translation with respect to the ground, the participation of all modes higher than the fundamental is zero.

The relationship between base isolated and partially restrained buildings is apparent. The frictionless base constitutes the perfect base isolation system. As the stiffness of the base isolation system diminishes, one would expect some features of partially restrained structures to show up in the dynamic behavior of the isolated buildings.

DYNAMICS OF MULTISTORY BASE ISOLATED BUILDINGS

Numerical studies of a number of base isolated buildings from 5 to 20 stories high, and either provided with moment frames, or shear walls, where performed in [4].The general conclusions on their dynamic behavior are similar to the ones derived from the 2 D.O.F. model described in the previous sections. The fundamental frequency has a low value, closely matching that one of a single degree of freedom system constituted by the building resting on isolators. The first mode shape is almost rectilinear, describes an almost uniform translation of the building as a whole, and approaches the vertical as the isolation stiffness decreases. The seismic response is strongly dominated by the first mode, with the participation factors of the higher modes being negligible. The effect of the isolation system on the frequencies and dampings, decreases as the order of the mode increases, but since the corresponding participation factors are very small, this fact is inconsequential.

The second mode along each principal direction of isolated buildings is quite sui-generis, and we could call it mode “1-Bis” to signify its close relationship to the quasi rigid body behavior of the first mode. It does not have a counterpart from the fixed base mode shapes. For the higher modes however, a correlation can be made between the isolated and non isolated modes, with the singularity that the isolated modes will always have a non zero base component.

Figure 7 shows schematic mode shapes for a three story small scale base isolated model, [8], where the mode 1-Bis and the discussed correlation are observed.

It would appear from the numerical studies in [4], that the softer the isolators, the less relevant is the response of the higher modes. It appears that the quasi rigid, almost vertical, fundamental mode shape induces this behavior. In effect, let us recall from the orthogonality properties that if such a mode is perfectly vertical, we can write that:

$$[1]^T MX_j = 0 \quad \text{for all } j>1 \tag{19}$$

Here M and K are the mass and stiffness matrices for the whole structure, and X_j is a mode of order j . The vector of inertia forces acting on the structure while vibrating in any mode is written:

$$F_i = -w_i^2 MX_i e^{iw_i t} \tag{20}$$

and the corresponding base shear is:

$$V_i = [1]^T F_i \tag{21}$$

that is:

$$V_i = -w_i^2 [1]^T MX_i e^{iw_i t} \tag{22}$$

but from equation (19), we have:

$$V = V_1 = -w_1^2 [1]^T M\{1\} e^{iw_1 t} = -w_1^2 M_T e^{iw_1 t} \tag{23}$$

where M_T is the total mass of the building, and:

$$V_j = 0 \quad \text{for all } j>1 \tag{24}$$

From the participation factor equation (8.a), we note that:

$$g_1 = 1 \tag{25}$$

and:

$$g_j = 0 \text{ for all j>1} \tag{26}$$

We conclude that if the base stiffness is sufficiently soft, the building will undergo a slow uniform translational motion. In the measure that this translation is uniform, the response of the building will be confined to the fundamental mode only, with the higher modes furnishing a negligible portion of the response. If in the limit, the base stiffness is zero and there is no friction between the base and the ground, we find the case of partially restrained structures in which the building stays still while the ground shakes horizontally beneath it. Note that a pure, truly vertical fundamental mode for isolated multistory buildings can only exist in partially restrained structures, or for perfectly rigid superstructures. In the last case the structure becomes in effect, a single degree of freedom system.

SEISMIC FORCES AND BASE DISPLACEMENTS

Returning to the two D.O.F. isolated structure, given that the participation factor for the second mode is very small, we confine our attention to the fundamental mode. As such, the subscripts denoting mode order are dropped. From equation (10), for practical purposes we may write:

$$V \cong V_b \tag{27}$$

that is, the isolation system damping is assumed to provide the total modal damping. In Appendix A, as well as in equation (23), it is demonstrated that the generalized mass for the superstructure of a properly base isolated building is approximately equal to its actual mass, therefore, from hereinafter M_S represents the total mass of the superstructure. Given a spectral pseudo acceleration A corresponding to the fundamental frequency, the seismic force applied to the base is:

$$F_b = g * A * M_b \tag{28}$$

Similarly, for the seismic force applied to the superstructure we write:

$$F_S = (1 + \Omega^2) * g * A * M_S \tag{29}$$

And the shear at ground level is simply:

$$V = F_b + F_S \tag{30}$$

The seismic forces acting on the superstructure may still be written:

$$F_S = g * A * M_S + g * \Omega^2 * A * M_S \tag{31}$$

The first term to the right of equation (31) represents forces associated with the “rigid body” portion of the response, whereas the second term represents forces linked to the flexibility of the superstructure. Note that this component is affected by the square of the frequential ratio, being therefore of reduced magnitude when compared to the “rigid” portion. The distribution of forces along the height of the building can now be made in proportion to the story masses for the “rigid” seismic force component. If the first mode shape of the building in its fixed base state is taken as rectilinear, a triangular distribution is obtained for the seismic forces associated with the flexibility of the building. Therefore we may write that for the mass M_i at height H_i above the base, the seismic force is:

$$F_i = g * A * M_i + \frac{M_i H_i}{\sum M_i H_i} * g * \Omega^2 * A * M_S \tag{32}$$

where the summation is carried to the number of stories above the base. The base displacements is found by dividing equation (30) by the isolation system stiffness:

$$x_b = \frac{g\,[m + (1+\Omega^2)]}{m+1} * \frac{A}{w_I^2} \tag{33}$$

If we assume that the fundamental frequency falls on the descending hyperbolic region of the pseudo acceleration spectra, corresponding to constant pseudo velocity, we may write:

$$A = \frac{A_I}{(1+\Omega^2)^{1/2}} \tag{34}$$

where A_I indicates the acceleration corresponding to a single degree of freedom system with frequency w_I, and equation (33) becomes:

$$x_b = x_{bI}\,\frac{g\,[m + (1+\Omega^2)]}{(m+1)(1+\Omega^2)^{1/2}} \tag{35}$$

where x_{bI} is the displacement for the single degree of freedom system with frequency w_I. For sufficiently small frequential ratios, we may finally write:

$$x_b = \frac{x_{bI}}{(1+\Omega^2)^{1/2}} \tag{36}$$

COMPARISON WITH UBC PROVISIONS

The Uniform Building Code [1], allows a simplified procedure for the design of base isolated buildings, which recognizes that the response of such structures is furnished mainly by the fundamental mode. Implicit in this procedure is the assumption of rigid body behavior for this mode, and hence, single degree of freedom structural behavior. Indeed, if the frequential ratio is small enough, equation (4) results in the code expression:

$$w \cong w_I \tag{37}$$

where the subscript I indicates single degree of freedom isolated building. Likewise, the code assumes that damping is provided by the isolation system only, per equation (27). With these assumptions, and given a pseudo acceleration response spectrum, the seismic forces and base displacements of the isolated structure are readily calculated.

Equation (36) is offered by the code as a more refined alternate way to calculate base displacements. It must be noticed however, that equation (36) is not provided to calculate base shears.

Regarding the distribution of lateral forces along the height of the building, the current code is inconsistent. While the provisions imply single degree of freedom behavior, assuming a rigid superstructure, they allow to distribute story forces according to a triangular distribution, which imply a flexible superstructure. This is an unwelcome change from the 1991 edition of the UBC which contained a distribution in proportion to the story weights, consistent with rigid body behavior, and hence provided a better approximation to the true distribution. A consistent, clearly more accurate, way of distributing these forces is given by equation (32).

The single degree of freedom assumption of the code will result in higher estimates of the fundamental frequencies and hence, higher shear forces as compared to the values obtained through a more accurate analysis. Consequently, the base displacements calculated per code will also be larger. Finally, for equal base shear values, the code procedure will always give higher story shears than more accurate analyses, in view of the inconsistency in the distribution of the story forces mentioned above.

CASE STUDY

Figure 3 shows a plane frame subjected to a horizontal seismic loading represented by the Uniform Building Code acceleration spectrum, Zone 4, and Soil Type 1. All modes are assumed to be damped with a coefficient of 5% of critical. The spectral dynamic analyses were carried out with SAP90 [3], which combines the modes using the Complete Quadratic Combination technique [5]. The calculated fixed base fundamental period is 0.97 seconds. The base horizontal flexibility was adjusted to represent a wide range of isolated structures. When considered a single degree of freedom, the building was provided with isolators resulting in fundamental periods ranging from 0.5 seconds to 3.5 seconds. Hence, the frequential ratio ranged from 0.277 to 1.94 .

Figure 4 shows the ratio of the fundamental period, as calculated with equation (4) (subscript c), to the “exact” fundamental period, as calculated with SAP90, plotted as a function of the reciprocal of the frequential ratio. Also plotted is the ratio of the fundamental period, as calculated with equation (37) (subscript i) per code procedure, to the “exact” period. The better approximation provided by equation (4) is obvious. The code procedure consistently underestimates the fundamental period, and only when the isolators are extremely flexible does the single degree of freedom assumption become reasonable. On the contrary, the fundamental period calculated with equation (4) gives a very good approximation of the correct period over the entire range explored.

Figure 5 shows the ratio of the base displacements, as calculated with equation (36) (subscript c), to the "exact" displacement, as calculated with SAP90, plotted as a function of the reciprocal of the frequential ratio. Similar plotting is shown for the ratio of the base displacements corresponding to a single degree of freedom (subscript i), per code procedure, to the "exact" displacement value. It is seen that the code procedure consistently over estimates base displacements, and only when the isolators are extremely flexible does the single degree of freedom approximation become acceptable. In contrast, the base displacements calculated with the procedure of this paper results in a very good approximation of the "exact" displacements over the whole range under study.

Figure 6 shows similar results for the base shears. Since the shears are equal to the base displacement times the isolation system stiffness, the same conclusions related to the base displacements apply to the base shears.

Table I shows the story shears for the frame isolated at a rigid body period of 2 second, which is a common isolation value for structures. The ratio of the shears obtained through the procedure of this paper (subscript c) and of the code procedure (subscript i), to the "exact" value per SAP90 is plotted as bar diagrams at each story. As shown, the code procedure consistently over estimates the story shears, while the procedure of this paper provides a more accurate shear distribution throughout the height of the structure. Nevertheless the shears calculated per the procedure of this paper are still slightly underestimated at the upper stories, very likely as a result of neglecting the higher modes.

CONCLUSIONS

A simple procedure has been developed to calculate the fundamental period, damping, base displacement, seismic base shear, and applied seismic story forces of base isolated buildings. The procedure is shown to yield the UBC equations when particularized to represent the behavior of single degree of freedom systems. A case study validates the procedure and shows an advantage when compared to the procedure of the UBC. The equations developed in this paper, in view of their simplicity and ease of application, provide a basis for future code procedures proposing a two degree of freedom base isolated model.

DISCLAIMER

The ideas and opinions expressed in this paper are solely the responsibility of the writer, and do not necessarily represent the policies and procedures of IDC.

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APPENDIX A

Let us consider the system depicted on figure 1.a. If excited at the base by a ground motion with horizontal acceleration *a(t)*, the equations of motion for the system are:

$$\begin{bmatrix} M & Mr \\ r^T M & M_b + r^T Mr \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y}_b \end{Bmatrix} + \begin{bmatrix} C & 0 \\ 0 & C_b \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y}_b \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & K_b \end{bmatrix} \begin{Bmatrix} x \\ y_b \end{Bmatrix} = -\begin{bmatrix} M & Mr \\ r^T M & M_b + r^T Mr \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a(t) \qquad \text{(A.1)}$$

where *r* is a vector of ones, and:

$$r^T Mr = \sum M_i$$

is the total mass of the superstructure. *K* and *C* are the superstructure’s stiffness and damping matrices. The *x’s* are the components of the vector of displacements relative to the base, and y_b is the displacement of the base relative to the ground. All other factors have been previously defined. If we make the assumption that the displacements of the superstructure relative to the base, may be written exclusively as a function of the fixed base first mode of the superstructure, that is:

$$\{x\} = x_S \{f\}$$

where x_S is the mode amplitude, and f is the fixed base mode shape. We may write:

$$\begin{Bmatrix} x \\ y_b \end{Bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_S \\ y_b \end{Bmatrix} \qquad \text{(A.2)}$$

Let us call the transformation matrix *A*, as follows:

$$[A] = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}$$

Replacing (A.2) into (A.1), and premultiplying by the transposed of *A*, we get:

$$\begin{bmatrix} f^T Mf & f^T Mr \\ r^T Mf & M_b + r^T Mr \end{bmatrix} \begin{Bmatrix} \ddot{x}_S \\ \ddot{y}_b \end{Bmatrix} + \begin{bmatrix} f^T Cf & 0 \\ 0 & C_b \end{bmatrix} \begin{Bmatrix} \dot{x}_S \\ \dot{y}_b \end{Bmatrix} + \begin{bmatrix} f^T Kf & 0 \\ 0 & K_b \end{bmatrix} \begin{Bmatrix} x_S \\ y_b \end{Bmatrix} = -\begin{bmatrix} f^T Mf & f^T Mr \\ r^T Mf & M_b + r^T Mr \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a(t) \qquad \text{(A.3)}$$

Let us transform the relative to base displacements, to relative to ground displacements, by means of the following transformation:

$$\begin{Bmatrix} x_S \\ y_b \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} y_S \\ y_b \end{Bmatrix} \qquad \text{(A.4)}$$

and let us call the transformation matrix B, as follows:

$$[B]=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$$

Replacing (A.4) into (A.3), and premultiplying by the transposed of *B*, we get equation (A.5):

$$\begin{bmatrix} f^T Mf & -f^T Mf+f^T Mr\\ -f^T Mf+r^T Mf & f^T Mf-2f^T Mr+M_b+r^T Mr\end{bmatrix}\begin{Bmatrix}\ddot{y}_S\\ \ddot{y}_b\end{Bmatrix}+\begin{bmatrix} f^T Cf & -f^T Cf\\ -f^T Cf & C_b+f^T Cf\end{bmatrix}\begin{Bmatrix}\dot{y}_S\\ \dot{y}_b\end{Bmatrix}+\begin{bmatrix} f^T Kf & -f^T Kf\\ -f^T Kf & K_b+f^T Kf\end{bmatrix}\begin{Bmatrix}y_S\\ y_b\end{Bmatrix}=$$

$$-\begin{bmatrix} f^T Mf & -f^T Mf+f^T Mr\\ -f^T Mf+r^T Mf & f^T Mf-2f^T Mr+M_b+r^T Mr\end{bmatrix}\begin{Bmatrix}1\\ 1\end{Bmatrix}a(t)$$

If the superstructure response is exclusively provided by the first mode, the participation factor for such mode must approach one. The participation factors for the higher modes become negligible. Under these circumstances we may write:

$$g=\frac{f^T Mr}{f^T Mf}\cong 1 \qquad \text{(A.6)}$$

In view of equation (A.6), the inertia matrices in equation (A.5) become diagonal. Additionally, the Effective Modal Mass, (E.M.M), provided by the first mode may be shown to be, [6]:

$$E.M.M.=\frac{\left[f^T Mr\right]^2}{f^T Mf} \qquad \text{(A.7)}$$

Inasmuch as the first mode of the superstructure, exclusively, provides the response, we can, in the limit, write:

$$f^T Mf\cong r^T Mr \qquad \text{(A.8)}$$

and equation (A.5) becomes:

$$\begin{bmatrix}M_S & 0\\ 0 & M_b\end{bmatrix}\begin{Bmatrix}\ddot{y}_S\\ \ddot{y}_b\end{Bmatrix}+\begin{bmatrix}C_S & -C_S\\ -C_S & C_S+C_b\end{bmatrix}\begin{Bmatrix}\dot{y}_S\\ \dot{y}_b\end{Bmatrix}+\begin{bmatrix}K_S & -K_S\\ -K_S & K_S+K_b\end{bmatrix}\begin{Bmatrix}y_S\\ y_b\end{Bmatrix}=-\begin{bmatrix}M_S & 0\\ 0 & M_b\end{bmatrix}\begin{Bmatrix}1\\ 1\end{Bmatrix}a(t) \qquad \text{(A.9)}$$

where:

$$M_S=f^T[M]f$$

$$C_S=f^T[C]f$$

$$K_S=f^T[K]f$$

are the generalized mass, damping, and stiffness respectively. Such is the equation of motion shown in the body of the text.

Equations (A.6), and (A.8) merit further discussion. For such equations to be fulfilled, the fixed base mode shape would have to approach a uniform translation, which is not generally the case. Hence equation (A.9) provides an approximation only. However, the dynamic properties of the superstructure are governed, and to some extent annulled, by the dynamic properties of the base isolated structure. Regarding the effective modal mass in particular, we found previously that the overall modal component j for the isolated superstructure is:

$$j = 1 + \Omega^2$$

We could write that at any level *i* the fixed base mode shape is a linear function of the square of the frequency ratio, i.e.:

$$j_i = 1 + a_i \Omega^2 \qquad \text{(A.10)}$$

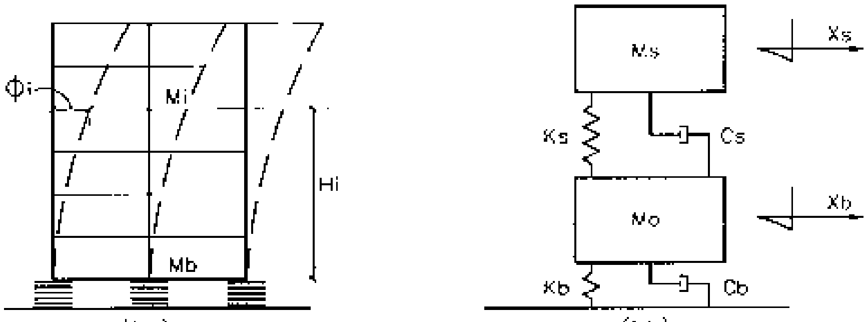
If the base mass of the isolated building is retained, the superstructure's equivalent mass is also given by equation (A.7) provided j is replaced for f. Replacing (A.10) we obtain:

$$E.M.M. = \sum M_i \, \frac{1 + \dfrac{2\Omega^2 \sum a_i M_i}{\sum M_i} + \Omega^4 \left(\dfrac{\sum a_i M_i}{\sum M_i} \right)^2}{1 + \dfrac{2\Omega^2 \sum a_i M_i}{\sum M_i} + \dfrac{\Omega^4 \sum a_i^2 M_i}{\sum M_i}}$$

where the summations carry over the number of levels above the base. If the terms to the fourth power of the frequential ratio are ignored, we have:

$$E.M.M. = \sum M_i \qquad \text{(A.11)}$$

Equation (A.11) indicates that the generalized mass for the superstructure of a base isolated building is very close to the actual building mass, and hence equation (A.9) is validated.

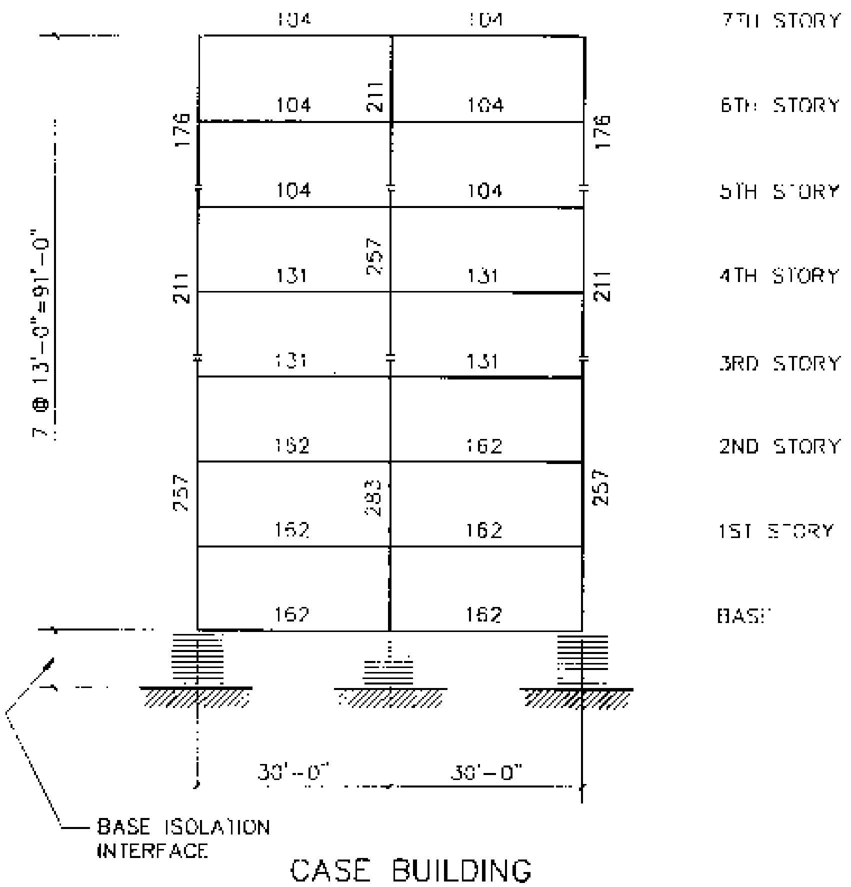


(1.a)　(1.b)

FIGURE 1

(2.c)　(2.b)

FIGURE 2



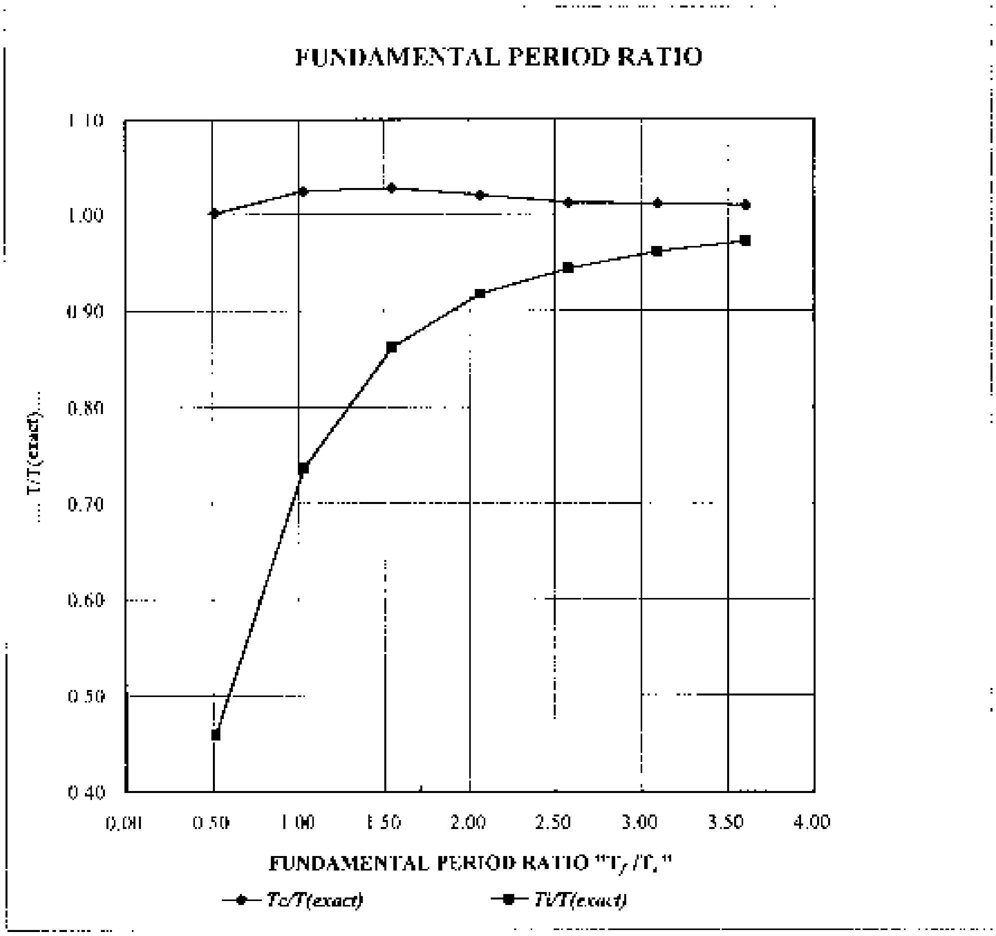
CASE BUILDING

TYPICAL STORY WEIGHT = 100 KIPS
MODULUS OF ELASTICITY = 29000 KSI

ALL COLUMNS ARE W14'S
ALL BEAMS ARE W24'S

MEMBER WEIGHTS ARE SHOWN

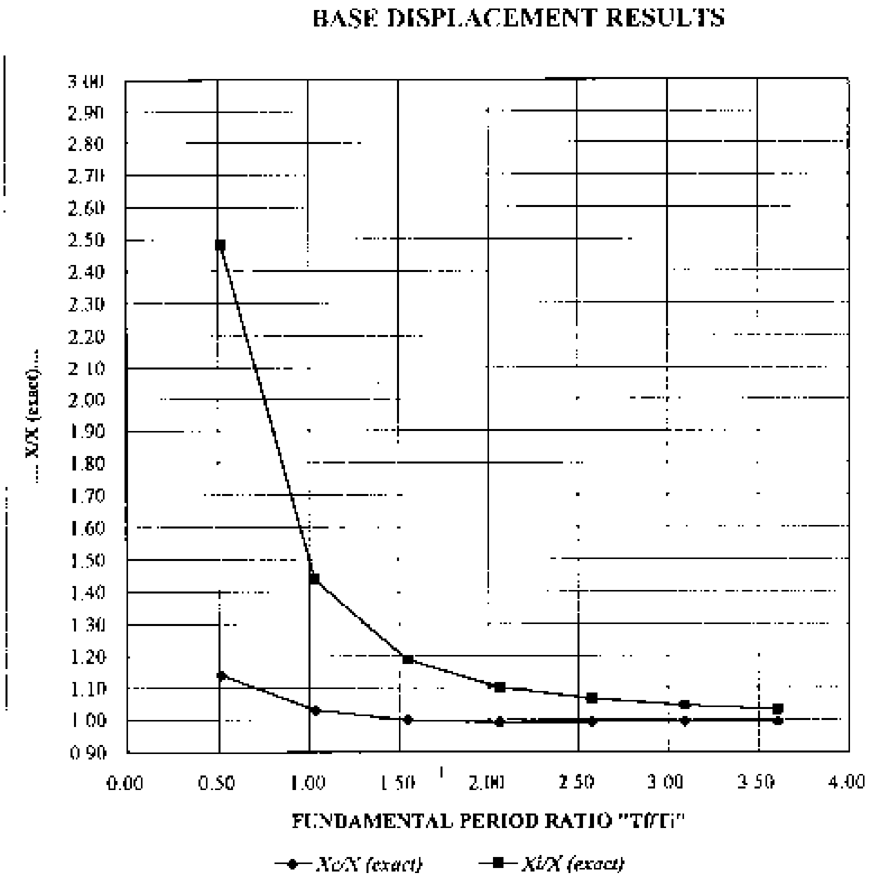
FIGURE 3



FUNDAMENTAL PERIOD RESULT (Tf=0.97)

CASES	EXACT SECOND	CALCULATED SECOND (Tc)	RIGID MODEL SECOND (Ti)	Tf/Ti
1	1.090	1.091	0.500	1.940
2	1.360	1.393	1.000	0.970
3	1.740	1.787	1.500	0.647
4	2.180	2.223	2.000	0.485
5	2.650	2.682	2.500	0.388
6	3.120	3.153	3.000	0.323
7	3.600	3.632	3.500	0.277

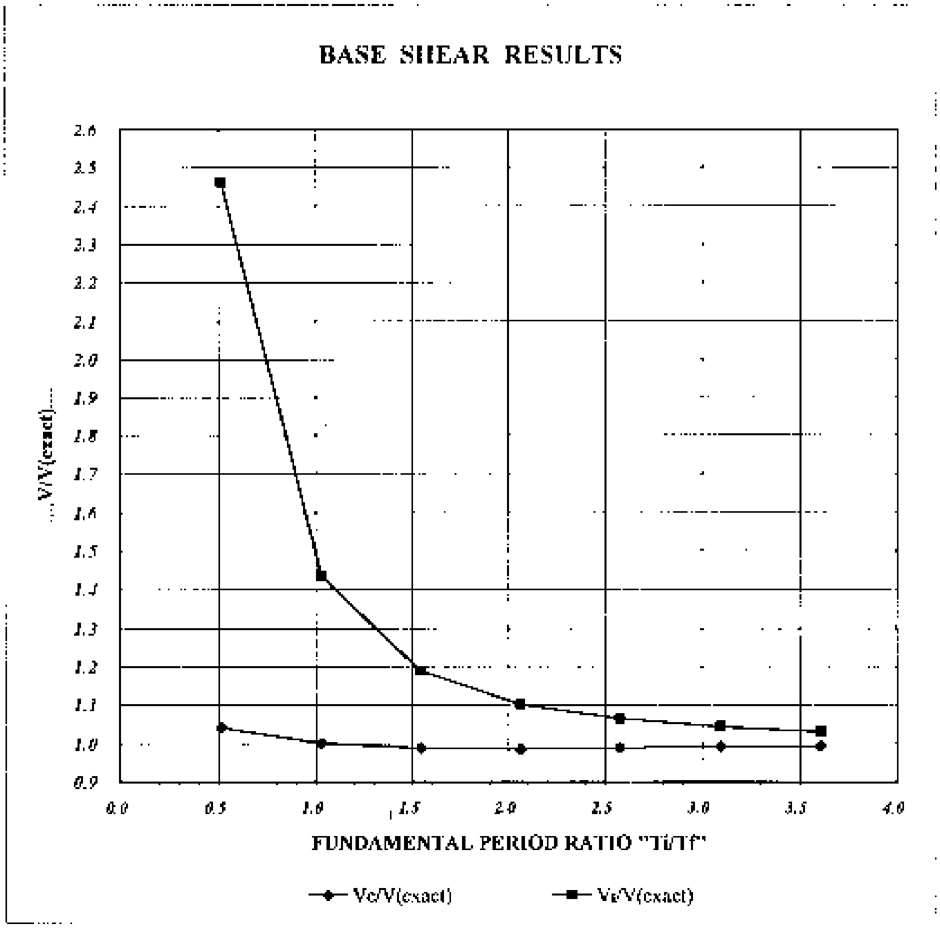
FIGURE 4



BASE DISPLACEMENTS

CASES	EXACT X	CALCULATED X_c	RIGID MODEL X_i	Tf/Ti
1	0.79	0.9	1.96	1.940
2	2.73	2.81	3.92	0.970
3	4.94	4.93	5.87	0.647
4	7.12	7.05	7.83	0.485
5	9.2	9.13	9.79	0.388
6	11.27	11.18	11.75	0.323
7	13.3	13.2	13.7	0.277

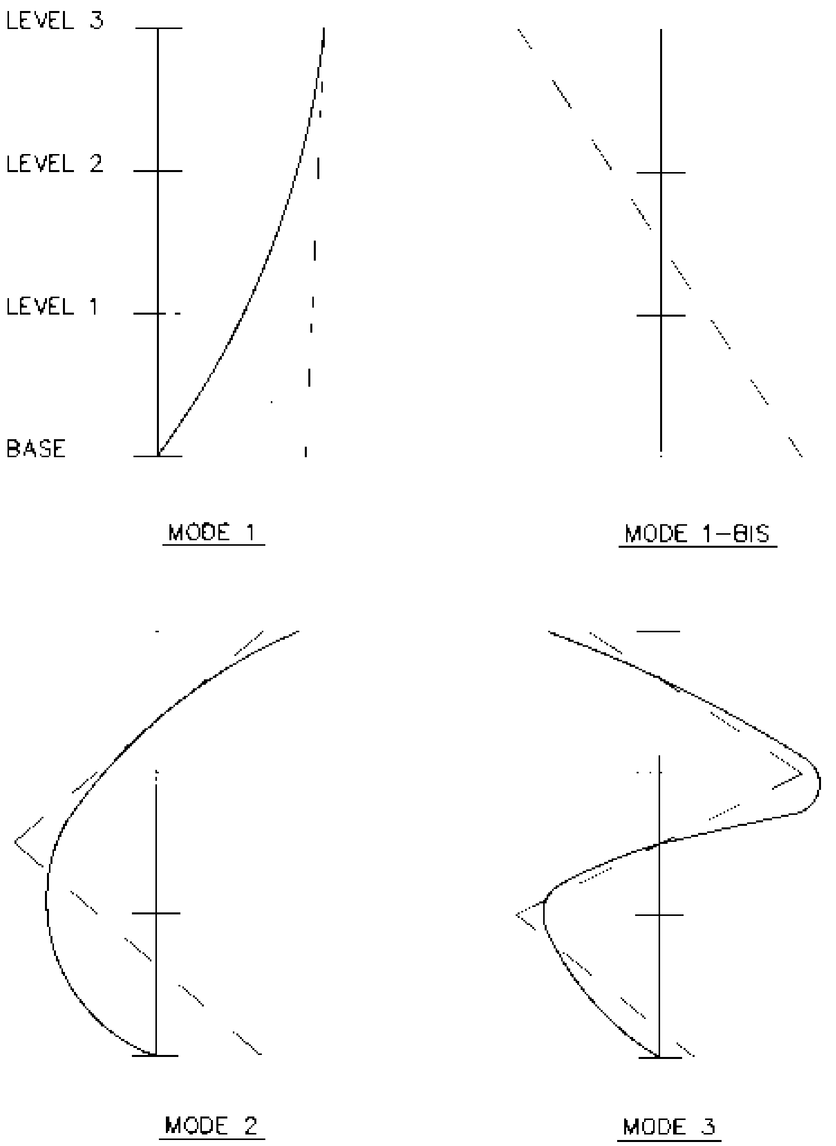
FIGURE 5



BASE SHEAR

CASES	EXACT KIPS	CALCULATED KIPS (Vc)	RIGID MODEL KIPS (Vl)	Tf/Tl
1	260.05	270.57	640	1.940
2	223.15	223.22	320	0.970
3	179.49	177.25	213.33	0.647
4	145.44	143.35	160	0.485
5	120.36	119.08	128	0.388
6	102.32	101.39	106.67	0.323
7	88.76	88.06	91.43	0.277

FIGURE 6



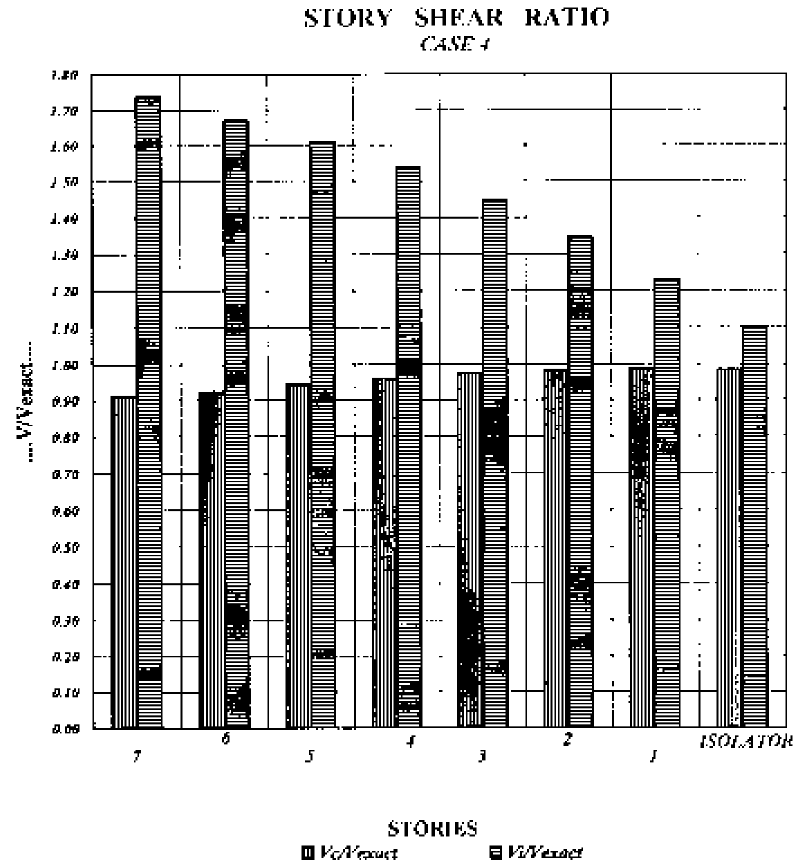
MODE 1 MODE 1-BIS

MODE 2 MODE 3

FIG. 7: EXPERIMENTAL MODE SHAPES WITH AND WITHOUT ISOLATORS.

— — = WITH ISOLATORS

——— = WITHOUT ISOLATORS



STORY SHEAR CASE 4

STORY	EXACT KIPS	CALCULATED KIPS (Vc)	UBC KIPS (VI)	V_C/Vexact	Vi/Vexact
7	23.03	20.98	40.00	0.911	1.737
6	44.46	41.08	74.29	0.924	1.671
5	63.94	60.31	102.86	0.943	1.609
4	81.86	78.67	125.72	0.961	1.536
3	98.62	96.15	142.86	0.975	1.449
2	114.62	112.75	154.29	0.984	1.346
1	130.15	128.49	160.00	0.987	1.229
ISOLATOR	145.44	143.35	160.00	0.986	1.100

TABLE 1